ELSEVIER

Journal of Geometry and Physics 28 (1998) 31–44

JOURNAL OF GEOMETRY AND PHYSICS

Holomorphic deformation of Hopf algebras and applications to quantum groups

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Received 17 March 1997; received in revised form 28 January 1998

Abstract

In this article we propose a new and so-called holomorphic deformation scheme for locally convex algebras and Hopf algebras. Essentially we regard converging power series expansions of a deformed product on a locally convex algebra, thus giving the means to actually insert complex values for the deformation parameter. Moreover we establish a topological duality theory for locally convex Hopf algebras. Examples coming from the theory of quantum groups are reconsidered within our holomorphic deformation scheme and topological duality theory. It is shown that all the standard quantum groups comprise holomorphic deformations. Furthermore we show that quantizing the function algebra of a (Poisson) Lie group and quantizing its universal enveloping algebra are topologically dual procedures indeed. Thus holomorphic deformation theory seems to be the appropriate language in which to describe quantum groups as deformed Lie groups or Lie algebras.

Subj. Class.: Quantum groups
1991 MSC: 81R50; 16W30; 46H
Keywords: Hopf algebras; Holomorphic deformation; Quantum groups

0. Introduction

In this paper we propose a new deformation scheme which we call holomorphic and which seems to recapture what is actually done in the context of describing quantum groups as deformed Lie algebras. Although it appears to be new as an explicitly formulated concept,

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0393-0440/98/$19.00
PII: S0393-0440(98)00007-2

we are convinced that our holomorphic deformation theory is in fact very close to many aspects of existing deformation procedures in mathematical physics.

The reason to consider holomorphic deformations instead of the by now classical formal deformations of Gerstenhaber (cf. [5,6]) is twofold. First, one likes to obtain concrete deformations, i.e. deformations of the structure on a given vector space which are defined on this vector space and not only on a suitable extension. Secondly, it is a well-known fact within the theory of infinite-dimensional Hopf algebras that one is often forced to change the usual tensor product and/or the concept of the dual space. This change is well understood by introducing a (locally convex) topology on the Hopf algebra in question as has been shown in [1]. There nuclear Hopf algebras H are studied, and a deformation of H is defined as a certain Hopf algebra structure on $H_T := \mathbb{C}[[T]]\hat{\otimes}H$, where $\hat{\otimes}$ denotes the completed π-tensor product of locally convex spaces. The change of definition we propose is simply to replace $\mathbb{C}[[T]]$ by the locally convex algebra $\mathcal{O}(\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$: A holomorphic deformation of H thus is a certain Hopf algebra structure on $H_\Omega := \mathcal{O}(\Omega)\hat{\otimes}H$. The advantage of this approach lies, among other things, in the fact that it is possible to actually insert values $z \in \Omega$ into the holomorphic deformation in order to get a deformed Hopf algebra structure on H (and not merely on H_T resp. H_Ω). Furthermore, the structure maps of our concrete deformations of H are evaluations of mappings depending holomorphically on z.

Of course, in order to show that this variation of a deformation concept is reasonable and useful, one has to give interesting examples. In the present paper we will show that a large part of the actually studied deformations of Hopf algebras, in particular those arising in the context of quantum groups, can in fact be interpreted as being holomorphic deformations.

Our work is motivated by the desire to understand physicists work on deformation quantization and inverse scattering, where algebras are "concretely deformed" by a real parameter $\hbar$ or q and not only by a formal one.

1. Topological and algebraic structures

Let us denote by $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$ together with the Euclidean topology. In order to formulate our concept of holomorphic deformation we need some preliminaries from the theory of locally convex and nuclear $\mathbb{K}$-vector spaces (cf. [10,11] for details). It is always assumed that such a locally convex (or briefly lc) topology is Hausdorff and complete.

Now, if E and F are two lc spaces, we denote the completion of the tensor product $E \otimes F$ endowed with the π-topology by $E\hat{\otimes}F$. An lc space E is called *nuclear* if all topologies on $E \otimes F$ compatible with $\otimes$ agree for all lc spaces F (cf. [7]). In case E and F are nuclear, the completion $E\hat{\otimes}F$ of $E \otimes F$ for two nuclear spaces E and F is again a nuclear space. We call a nuclear space E *strictly nuclear* if its (strong) dual E' is nuclear as well, if E is reflexive (i.e. the strong dual of E' is isomorphic to E as an lc space) and if it fulfills the *duality condition*, i.e. the canonical linear mapping $E' \otimes E' \to (E\hat{\otimes}E)'$ extends to an algebraic and topological isomorphism $E'\hat{\otimes}E' \to (E\hat{\otimes}E)'$. Nuclear Fréchet spaces or duals of nuclear Fréchet spaces are strictly nuclear as well as nuclear LF-spaces.

Algebraic structures can now be formulated within the symmetric monoidal category of lc vector spaces with $\hat{\otimes}$ as tensor product.

Definition 1.1. An *lc algebra* is an lc space A together with continuous linear mappings $\mu : A\hat{\otimes}A \to A$ and $\eta : \mathbb{K} \to A$ such that μ fulfills the associativity constraint, and η gives rise to a unit. A homomorphism between lc algebras A and $\tilde{A}$ is just a continuous linear map $f : A \to \tilde{A}$ such that $\tilde{\mu} \circ (f\hat{\otimes}f) = f \circ \mu$ and $\tilde{\eta} \circ f = \eta$. A *locally m-convex algebra* is an lc algebra A for which there exists a defining family of multiplicative seminorms (cf. [9]). A *nuclear algebra* (resp. a *strictly nuclear algebra*) is an lc algebra for which the underlying lc space A is a nuclear space (resp. a strictly nuclear space).

Similarly one defines the concepts of *lc coalgebra* (resp. *lc bialgebra* and *lc Hopf algebra*): These are lc spaces together with continuous structure maps fulfilling the appropriate structure axioms with $\hat{\otimes}$ as tensor product functor. Moreover, by a morphism of lc coalgebras (resp. lc bialgebras and lc Hopf algebras), we understand a continuous linear map leaving the structure maps invariant. After these definitions it is obvious what is meant by a (strictly) nuclear coalgebra and so on.

Remark 1.2. An lc space A comprises an lc algebra if and only if it has an underlying structure of a $\mathbb{K}$-algebra such that the multiplication $\mu : A \times A \to A$ is continuous. An analogous result does not hold for coalgebras. Namely there exist lc coalgebras C which do not have an underlying structure of a coalgebra. In other words this means that the map $\Delta : C \to C\hat{\otimes}C$ need not have its image in $C \otimes C$. An example is given by the quantized $\mathfrak{sl}(N+1, \mathbb{C})$ of Section 3.

Definition 1.3. Let H be an lc Hopf algebra or bialgebra. It is called *topologically quasitriangular* if there exists an invertible element $\mathcal{R} \in H\hat{\otimes}H$ such that the following conditions hold:

$$\tau \circ \Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{1}$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2}$$

where $\tau : H\hat{\otimes}H \to H\hat{\otimes}H$ is the flip-morphism, and $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, and $\mathcal{R}_{23}$ are the obvious extensions of $\mathcal{R}$ to $H\hat{\otimes}H\hat{\otimes}H$ which are trivial on the third, second and first factor, respectively. In this case $\mathcal{R}$ is called the *topological universal R-matrix* of H. If additionally $\mathcal{R}^{-1} = \tau \circ \mathcal{R}$, then H is called *topologically triangular*. Dually H is called *topologically coquasitriangular* if there exists a continuous bilinear map $\langle\ |\ \rangle : H\hat{\otimes}H \to \mathbb{C}$, the *braiding form*, such that for all $a, b, c \in H$

$$\sum_{(a),(b)} \langle a_1|b_1\rangle a_2b_2 = \sum_{(a),(b)} a_1b_1\langle a_2|b_2\rangle, \tag{3}$$

$$\langle a|bc\rangle = \sum_{(a)}\langle a_1|b\rangle\langle a_2|c\rangle, \qquad \langle ab|c\rangle = \sum_{(c)}\langle a|c_1\rangle\langle b|c_2\rangle. \tag{4}$$

One of the main reasons to consider strictly nuclear Hopf algebras instead of just Hopf algebras lies in the fact that the category of strictly nuclear Hopf algebras has duals.

Proposition 1.4. *Let H be a strictly nuclear Hopf algebra. Then H' carries in a natural way the structure of a nuclear Hopf algebra. Moreover, H can be recovered as H''. If H is topologically quasitriangular (resp. coquasitriangular) then H' is topologically coquasitriangular (resp. quasitriangular). The same holds for reflexive lc Hopf algebras fulfilling the duality condition.*

Proof. For the first part of the proposition just apply the isomorphism $(H\hat{\otimes}H)' \cong H'\hat{\otimes}H'$ to obtain the coproduct Δ' on H' as the pull-back $\mu^* : H' \longrightarrow (H\hat{\otimes}H)', f \longmapsto f \circ \mu$. The other structure maps of H' are directly defined by transposition. For the proof of the second part note that an element $\mathcal{R} \in H\hat{\otimes}H$ induces a continuous bilinear form $\langle \mid \rangle_{\mathcal{R}} : H'\hat{\otimes}H' \to \mathbb{C}$ by $f \otimes g \mapsto f \otimes g(\mathcal{R})$. Then, by the isomorphism $(H\hat{\otimes}H)' \cong H'\hat{\otimes}H'$ every continous bilinear form $\langle \mid \rangle : H \otimes H \to \mathbb{C}$ can be interpreted as an element $\mathcal{R}_{\langle \mid \rangle} \in H'\hat{\otimes}H'$. The proof of the required algebraic properties for the thus defined braiding form $\langle \mid \rangle_{\mathcal{R}}$ resp. R-matrix $\mathcal{R}_{\langle \mid \rangle}$ follows exactly like in the well-known finite-dimensional case. □

2. Natural locally convex and nuclear Hopf algebras

In this section we will consider some topological constructions and examples of lc Hopf algebras.

2.1. *Inductive limit topologies on Hopf algebras*

On a given Hopf algebra H over $\mathbb{K}$ we can always consider the finest lc topology. Then H is a complete lc Hopf algebra, since all structure maps are automatically continuous for the finest locally convex topology. The locally convex Hopf algebra H is locally m-convex only if H is finite-dimensional and nuclear only if H is of countable dimension. The dual H' with the strong topology carries the coarsest lc topology. Since $H \cong \mathbb{K}^{(\Lambda)}$ satisfies the duality condition, the dual H' is a lc Hopf algebra as well according to Proposition 1.4. For the special case of a group algebra $H = \mathbb{K}G$ this dual is the Hopf algebra $H' \cong \mathbb{K}^G$ of all functions on the group G.

2.2. *Projective limit topologies on Hopf algebras*

Alternatively one can provide a given Hopf algebra H over $\mathbb{K}$ with the lc projective limit *topology of finite-dimensional representations*, i.e. with the coarsest lc topology leaving continuous all finite-dimensional representations $\varphi : H \to \operatorname{End} V$. Let us assume that these representations separate the points of H. According to the later proved Proposition 2.1 this is the case for example for finitely generated Hopf algebras H. It also holds for the universal enveloping algebra $H = \mathcal{U}\mathfrak{g}$ of a finite-dimensional Lie algebra $\mathfrak{g}$. Then H is a nuclear Hausdorff locally convex space which is complete only if H is finite-dimensional. All structure maps of the Hopf algebra H are continuous since the topology is adapted to the finite-dimensional representations. Therefore, they can be uniquely extended to the completion $\hat{H}$ of H and thus turn $\hat{H}$ into a nuclear Hopf algebra. In addition $\hat{H}$ is locally m-convex. The dual $\hat{H}'$ of $\hat{H}$ (or of H) is the space of matrix coefficients on H, i.e. $\hat{H}' = \{\xi \circ \hat{\varphi} | \varphi : H \to \mathrm{End}\, V$ finite-dimensional representation, $\xi \in (\mathrm{End}\, V)'\}$. Hence, $\hat{H}'$ coincides with the restricted dual H° of H (see [2, Chapter 4.1.D]). The strong topology on $\hat{H}'$ is given by the locally convex inductive limit topology of the maps $\varphi' : (\mathrm{End}\, V)' \to \hat{H}'$, $\xi \mapsto \xi \circ \hat{\varphi}$, where φ runs through all finite-dimensional representations of H. Thus the strong topology on $\hat{H}'$ is the finest lc topology. Although the lc space $\hat{H}' = H^\circ$ is in general not nuclear, it satisfies the duality condition. Therefore, as in Proposition 1.4 the transpositions of the structure maps of the lc Hopf algebra $\hat{H}$ define the structure of an lc Hopf algebra on $\hat{H}'$. Dualizing again one gets the Hausdorff completion $H^{\circ\prime}$ of H. The lc space $\hat{H}'$ will be strictly nuclear, whenever countably many of the finite-dimensional representations generate the topology of H, i.e. if $\hat{H}$ is Fréchet.

In the same spirit one can consider other projective systems of representations of H in order to define appropriate locally convex topologies on H. For example H can be endowed with the projective limit topology of all homomorphisms $H \to A$, where A is a nuclear locally m-convex Fréchet algebra. We call the resulting projective limit topology the *topology of nuclear Fréchet representations*. The completion $\check{H}$ of H with respect to this topology again is a nuclear Hopf algebra. Furthermore, one has a natural continuous inclusion $\check{H} \to \hat{H}$.

Proposition 2.1. *Let A be a finitely generated algebra and $\mathrm{T}V \xrightarrow{\pi} A$ a presentation of A, where $\mathrm{T}V$ is the tensor algebra of a finite-dimensional $\mathbb{K}$-vector space V. Denote by $\hat{\mathrm{T}}V$ and $\hat{A}$ (resp. $\check{\mathrm{T}}V$ and $\check{A}$) the completions of $\mathrm{T}V$ and A with respect to the topology of finite-dimensional (resp. nuclear Fréchet) representations. Then the algebras $\hat{\mathrm{T}}V$, $\hat{A}$, $\check{\mathrm{T}}V$ and $\check{A}$ are locally m-convex nuclear Hausdorff spaces. The spaces $\check{\mathrm{T}}V$ and $\check{A}$ are even Fréchet. Furthermore, the presentation $\mathrm{T}V \xrightarrow{\pi} A$ extends uniquely to surjective and open maps $\hat{\mathrm{T}}V \xrightarrow{\hat{\pi}} \hat{A}$ and $\check{\mathrm{T}}V \xrightarrow{\check{\pi}} \check{A}$, i.e. to topological presentations of $\hat{A}$ and $\check{A}$.*

Proof. By the universal property of the complete hull we have unique morphisms $\hat{\mathrm{T}}V \xrightarrow{\hat{\pi}} \hat{A}$ and $\check{\mathrm{T}}V \xrightarrow{\check{\pi}} \check{A}$ both extending $\mathrm{T}V \xrightarrow{\pi} A$. We will show that they have the claimed properties. First consider the topology of finite-dimensional representations. Let us show that this topology is Hausdorff or in other words that the finite-dimensional representations of A separate the points of A. Consider the ideals $I_n = \bigoplus_{k \geq n} V^{\oplus k}$ in $\mathrm{T}V$. Their images in A define ideals J_n in A. Now for two elements $a, b \in A$, $a \neq b$, there exists an $n \in \mathbb{N}$ large enough such that $a - b$ does not vanish in A/J_n. But A/J_n is finite-dimensional and an A-module. Thus the points a and b are separated by the representation of A on A/J_n. The continuous homomorphism $\hat{\mathrm{T}}V \to \hat{A}$ is an open map. To see this choose an open set $U \subset \hat{\mathrm{T}}V$. We can assume that there exists a finite-dimensional representation $\varphi : \hat{\mathrm{T}}V \to \mathrm{End}\, W$ and an open $O \subset \mathrm{End}\, W$ such that $U = \varphi^{-1}(O)$. Let $\hat{I}$ be the kernel of $\hat{\pi}$, and $\tilde{W}$ be the algebra $\mathrm{im}\, \varphi / \varphi(\hat{I})$. Then φ induces a representation $\tilde{\varphi} : A \to \tilde{W} \subset \mathrm{End}\, \tilde{W}$. As projections between finite-dimensional spaces are open there exists an open $\tilde{O} \subset \mathrm{End}\, \tilde{W}$ such that $\tilde{O} \cap \mathrm{im}\, \tilde{\varphi} = O + \varphi(\hat{I})$. We then have $\hat{\pi}(U) = \tilde{\varphi}^{-1}(\tilde{O})$, i.e. $\hat{\pi}(U)$ is open in $\hat{A}$.

As $\hat{\mathrm{T}}V \to \hat{A}$ is continuous, open and has dense image, it is surjective, hence, provides a continuous presentation of $\hat{A}$.

The proof for the case of nuclear Fréchet representations goes along similar lines. □

2.3. *Matrix coefficients of group representations*

For a group G consider the Hopf algebra of complex-valued matrix coefficients $\mathcal{R}_0(G) := \{\xi \circ \varphi | \varphi : G \to \mathrm{GL} V$ finite-dimensional complex representation, $\xi \in (\mathrm{End}\, V)'\}$ in the light of the preceding two examples. The space $\mathcal{R}_0(G)$ with the finest lc topology is an lc Hopf algebra fulfilling the duality condition. This topology can also be described as the lc inductive limit of the maps $\varphi' : (\mathrm{End}\, V)' \to \mathcal{R}_0(G)$ where φ runs through all finite-dimensional representations of G. The dual $\mathcal{R}_0(G)'$ of $\mathcal{R}_0(G)$ endowed with the strong topology is an lc projective limit of finite-dimensional algebras and thus is a nuclear Hopf algebra which is locally m-convex. By the duality condition for $\mathcal{R}_0(G)$ the dual $\mathcal{R}_0(G)'$ obtains as in Proposition 1.4 the structure of a nuclear Hopf algebra.

In case of a topological group G we replace $\mathcal{R}_0(G)$ by the continuous matrix coefficients $\mathcal{R}(G) \subset \mathcal{R}_0(G)$. $\mathcal{R}(G)$ with the finest lc topology is an lc Hopf algebra as well and the dual $\mathcal{R}(G)'$ is a nuclear locally m-convex algebra. In general, $\mathcal{R}(G)$ is not nuclear. For compact groups, however, $\mathcal{R}(G)$ is strictly nuclear, since by the theorem of Peter and Weyl it is of countable dimension. Moreover, the dual $\mathcal{R}(G)'$ is a Fréchet nuclear locally m-convex algebra.

2.4. *Universal enveloping algebras*

Starting with a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is a Hopf algebra over $\mathbb{K}$. Then the topology of finite dimensional representations on $\mathcal{U}\mathfrak{g}$ is Hausdorff, and the completion $\hat{\mathcal{U}}\mathfrak{g}$ of $\mathcal{U}\mathfrak{g}$ is a nuclear Hopf algebra. Moreover, $\hat{\mathcal{U}}\mathfrak{g}$ is locally m-convex.

For a Lie group G with Lie algebra $\mathfrak{g}$ there is a natural map i relating $\hat{\mathcal{U}}\mathfrak{g}$ and the nuclear Hopf algebra $\mathcal{R}(G)'$. The map $i : \mathcal{U}\mathfrak{g} \to \mathcal{R}(G)'$ is defined by $i(X)(f) = L_X f(e)$, $f \in \mathcal{R}(G)$, where L_X is the left invariant differential operator on G given by $X \in \mathcal{U}\mathfrak{g}$, and e is the unit of G. Now i can be extended to a continuous $\mathbb{R}$-linear map $i : \hat{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ which is a morphism of nuclear Hopf algebras. In general i is not injective, but for connected and simply connected Lie groups G it is. Moreover, in that case i is an open map onto its image $i(\hat{\mathcal{U}}\mathfrak{g})$, since the finite-dimensional complex representations of $\mathfrak{g}$ and G are in one-to-one correspondence. Therefore, $\hat{\mathcal{U}}\mathfrak{g}$ can be considered as a closed nuclear sub-Hopf algebra of $\mathcal{R}(G)'$.

2.5. *Simple Lie algebras*

If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ then finitely many of the finite-dimensional representations already generate all finite-dimensional representations of $\mathfrak{g}$ (via finite sums and tensor products; e.g. for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})$ the finite-dimensional representations are generated by the fundamental representation $\mathfrak{sl}(N, \mathbb{C}) \subset \mathfrak{gl}(N, \mathbb{C})$). As a consequence, the topology of finite-dimensional representations on the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is metrizable, hence the completion $\hat{\mathcal{U}}\mathfrak{g}$ is Fréchet. Therefore $\hat{\mathcal{U}}\mathfrak{g}$ is strictly nuclear.

For a simple complex Lie algebra $\mathfrak{g}$ with corresponding connected and simply connected Lie group G the image $i(\mathcal{U}\mathfrak{g})$ under the map $i : \hat{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ is dense in $\mathcal{R}(G)'$. Hence, by the above, the map $i : \hat{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ is an isomorphism of Fréchet algebras. Thus we have a natural complete duality between $\hat{\mathcal{U}}\mathfrak{g}$ and $\mathcal{R}(G)'$. In particular, $\hat{\mathcal{U}}\mathfrak{sl}(N, \mathbb{C}) \cong \mathcal{R}(\mathrm{SL}(N, \mathbb{C})'$.

In the case of a compact Lie group G the map $i : \mathcal{U}\mathfrak{g} \to \mathcal{R}(G)'$ is injective as well. It can be continued to a $\mathbb{C}$-linear injective map $i : \mathcal{U}\mathfrak{g}^{\mathbb{C}} \to \mathcal{R}(G)'$ by complexification of the Lie algebra $\mathfrak{g}$. The image $i(\mathcal{U}\mathfrak{g}^{\mathbb{C}})$ turns out to be dense in $\mathcal{R}(G)'$. However, the induced topology on $i(\mathcal{U}\mathfrak{g}^{\mathbb{C}}) \subset \mathcal{R}(G)'$ does in general not coincide with the topology coming from the projective topology of finite-dimensional representations; see e.g. the example of $U(1)$. Instead of this, the inclusion i induces a new lc topology on $\mathcal{U}\mathfrak{g}^{\mathbb{C}}$ which can be described as the lc projective limit of all representations $\dot{\varphi}$ which are derivatives of finite-dimensional continuous representations φ of the group G. This topology depends on the group in question and not only on the Lie algebra $\mathfrak{g}$. It is always metrizable and nuclear. Hence, the completion – which we denote by $\tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$ – is a Fréchet nuclear Hopf algebra naturally isomorphic to $\mathcal{R}(G)'$.

As the completion $\check{\mathcal{U}}\mathfrak{g}$ of $\mathcal{U}\mathfrak{g}$ with respect to the topology of nuclear Fréchet representations naturally lies in $\hat{\mathcal{U}}\mathfrak{g}$ we have a continuous inclusion $\check{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ as well. Note that $\hat{\mathcal{U}}\mathfrak{g}, \check{\mathcal{U}}\mathfrak{g}, \tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$ and $\mathcal{R}(G)'$ are locally m-convex algebras as projective limits of locally m-convex algebras.

3. Holomorphic deformation of locally convex algebras

In our approach to deformation theory we replace the ring $\mathbb{C}[[T]]$ used in the formal deformation theory of algebras (cf. [1,5]) by the nuclear Fréchet algebra $\mathcal{O}(\Omega)$ of holomorphic functions on an open complex domain $\Omega \subset \mathbb{C}^n$. For a complete lc Hausdorff space E, let $E_\Omega = \mathcal{O}(\Omega, E)$ be the space of holomorphic E-valued functions $f : \Omega \to E$ equipped with the compact open topology. Every E can locally be represented by a convergent power series $f(z) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha$, $z \in \Omega$, where $f_\alpha \in E$. Now E_Ω is a complete lc Hausdorff space which is isomorphic to the completion of the tensor product $\mathcal{O}(\Omega) \otimes E$ with respect to the π-topology (cf. [7]):

$$E_\Omega = \mathcal{O}(\Omega, E) \cong \mathcal{O}(\Omega) \hat{\otimes} E. \tag{5}$$

Hence, E_Ω is nuclear (resp. Fréchet) if E is nuclear (resp. Fréchet). The pointwise multiplication $\mathcal{O}(\Omega) \times E_\Omega \to E_\Omega$, $(\lambda, f) \mapsto \lambda f$, is continuous and thus defines on E_Ω the structure of a *topological* $\mathcal{O}(\Omega)$*-module*. Because of Eq. (5) we sometimes call E_Ω a *topologically free* $\mathcal{O}(\Omega)$*-module*. If E and F are topological $\mathcal{O}(\Omega)$-modules, we denote by $E \hat{\otimes}_{\mathcal{O}(\Omega)} F$ the completion of $E \otimes_{\mathcal{O}(\Omega)} F$ by the π-topology.

In deformation theory one is interested in algebra structures on E_Ω. Let us describe this in more detail. Define a *topological* $\mathcal{O}(\Omega)$*-algebra structure* on E_Ω to be an algebra structure on $E_\Omega = \mathcal{O}(\Omega, E)$ given by a continuous $\mathcal{O}(\Omega)$-bilinear map $\tilde{\mu} : E_\Omega \times E_\Omega \longrightarrow E_\Omega$ fulfilling the associativity constraint and a continuous $\mathcal{O}(\Omega)$-linear unit $\tilde{\eta} : \mathcal{O}(\Omega) \longrightarrow E_\Omega$. We often denote this algebra $(E_\Omega, \tilde{\mu}, \tilde{\eta})$ by $\tilde{E}$. $\tilde{\eta}$ is determined by $\tilde{\eta}(1) \in E_\Omega$.

Two such algebra structures $(\tilde{\mu}, \tilde{\eta})$ and $(\check{\mu}, \check{\eta})$ on E_Ω are called *equivalent* if there exists an $\mathcal{O}(\Omega)$-linear isomorphism $\varphi : E_\Omega \to E_\Omega$ (of lc spaces) such that the relations $\varphi \circ \check{\mu} = \tilde{\mu} \circ (\varphi \times \varphi)$ and $\varphi \circ \check{\eta} = \tilde{\eta}$ are fulfilled.

The multiplication $\tilde{\mu}$ of an $\mathcal{O}(\Omega)$-algebra $(E_\Omega, \tilde{\mu}, \tilde{\eta})$ can be described as a holomorphic map $\tilde{\mu} : \Omega \to \mathcal{L}(E\hat{\otimes}E, E)$, where $\mathcal{L}(E, F)$ is the space of continuous linear maps from E to F equipped with the topology of uniform convergence on the bounded sets of E. Thus around any $z_0 \in \Omega$ the product map $\tilde{\mu}$ can locally be expanded in the form

$$\tilde{\mu}(z) = \sum_{\alpha \in \mathbb{N}^n} \mu_\alpha (z - z_0)^\alpha, \tag{6}$$

where $z \in \Omega$ is close enough to z_0 and the μ_α are continuous bilinear mappings on E with values in E and which depend on the base point z_0. Similarly $\tilde{\eta} : \mathbb{C} \to E_\Omega$ with $\tilde{\eta}(\lambda) = \lambda\tilde{\eta}(1)$ can be viewed as the map $\tilde{\eta}(1) : \Omega \to E$. In case $E = A$ is an lc algebra the *constant algebraic structure* on A_Ω is given by $\tilde{\mu}(z) = \mu$ and $\tilde{\eta}(\lambda)(z) = \eta(\lambda)$, $\lambda \in \mathbb{C}$, $z \in \Omega$.

Definition 3.1. Let $\Omega \subset \mathbb{C}$ be an open complex domain, $*$ a distinguished point and $\mathfrak{m}_* \subset \mathcal{O}(\Omega)$ the maximal ideal of holomorphic vanishing at $*$. A *(topologically free) holomorphic deformation* of a lc algebra (A, μ, η) over Ω at $*$ is a topological $\mathcal{O}(\Omega)$-algebra structure $(\tilde{\mu}, \tilde{\eta})$ on A_Ω such that the quotient $\mathcal{O}(\Omega)$-module $A_\Omega/\mathfrak{m}_* A_\Omega$ is isomorphic to A as a lc algebra. Equivalently $\mu_0 = \mu$ in expansion (6) and $\tilde{\eta}(1)(*) = \eta(1)$. The deformation is called *trival* if $(A_\Omega, \tilde{\mu}, \tilde{\eta})$ is equivalent to the constant algebra structure on A_Ω. The distinguished point $*$ is called the *base point* of the deformation.

An important advantage of holomorphic deformations in comparison to formal deformations lies in the fact that for every parameter $z \in \Omega$ one receives a *concrete* deformed algebra structure on the underlying linear space of the original algebra A: Simply take – for any value $z \in \Omega$ – $\tilde{\mu}(z) \in \mathcal{L}(A\hat{\otimes}A, A)$ as the deformed multiplication and $\tilde{\eta}(z) \in \mathcal{L}(\mathbb{C}, A)$ as the deformed unit. Then $a *_z b = \tilde{\mu}(z)(a, b) \in A$ is the new product of $a \in A$ and $b \in A$, and $e_z = \tilde{\eta}(z)(1) \in A$ is the new unit. Both, the new product and the new unit are contained in the original space A and not only in a nontrivial extension of A.

In case the lc algebra A is commutative and carries a Poisson bracket $\{\ ,\ \}$ one is interested in the quantization of A or in other words deformations of A in direction of the Poisson bracket. More precisely under a *(topologically free) holomorphic quantization* of A over the domain $\Omega \subset \mathbb{C}$ we understand a topologically free holomorphic deformation $(A_\Omega, \tilde{\mu}, \tilde{\eta})$ of A at base point $0 \in \Omega$ such that the relation

$$\tilde{\mu}(f, g) - \tilde{\mu}(g, f) = -\mathrm{i}z\{f, g\} + \mathrm{o}(z^2) \tag{7}$$

holds for all $f, g \in A$ and $z \in \Omega$.

Example 3.2.

(i) *Quantum vector spaces* (cf. [4,8]). Let $\Omega = \mathbb{C}^*$ be the set of all nonzero complex numbers and consider the n-dimensional complex vector space $V = \mathbb{C}^n$ with the canonical basis $(x_1, \ldots, x_n)$. Then construct the tensor algebra TV of V or in other words the free $\mathbb{C}$-algebra in n generators. By completion with respect to the projective limit topology of all Fréchet representations TV becomes a nuclear locally m-convex Fréchet algebra $\check{\mathrm{T}}V$. The functions

$$f_{\iota\kappa} : \Omega \to \mathrm{T}V, \quad z \mapsto x_\iota x_\kappa - z x_\kappa x_\iota \quad 1 \le \iota < \kappa \le n \tag{8}$$

then generate a unique closed ideal I in the nuclear algebra $\mathcal{O}(\Omega)\hat{\otimes}\check{\mathrm{T}}V$. The quotient $\mathcal{O}(\mathbb{C}^n_q) = \mathcal{O}(\Omega)\hat{\otimes}\check{\mathrm{T}}V/I$ is called the *algebra of entire functions on the quantum n-vector space*. It comprises a holomorphic deformation of the algebra $\mathcal{O}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$: The homomorphism $\mathcal{O}(\mathbb{C}^n_q) \to \mathcal{O}(\mathbb{C}^n)$ defined by $[f \otimes x_\iota] \mapsto f(1)x_\iota$ is well-defined, surjective and has kernel $\mathfrak{m}_1\mathcal{O}(\mathbb{C}^n_q)$. Therefore $\mathcal{O}(\mathbb{C}^n_1)/\mathfrak{m}_1\mathcal{O}(\mathbb{C}^n_q) \cong \mathcal{O}(\mathbb{C}^n)$ holds. Because of relations (8) the $\mathcal{O}(\mathbb{C}^*)$-linear combinations of the family $(x_1^{m_1}, \ldots, x_n^{m_n})_{m_1,\ldots,m_n \in \mathbb{N}}$ are dense in $\mathcal{O}(\mathbb{C}^n_q)$. Since $(x_1^{m_1}, \ldots, x_n^{m_n})_{m_1,\ldots,m_n \in \mathbb{N}}$ furthermore is free over $\mathcal{O}(\mathbb{C}^*)$, $\mathcal{O}(\mathbb{C}^n_q)$ is isomorphic to $\mathcal{O}(\mathbb{C}^*)\hat{\otimes}\mathcal{O}(\mathbb{C}^n) \cong \mathcal{O}(\mathbb{C}^* \times \mathbb{C}^n)$ as a nuclear space. This proves the claim.

Alternatively one could give TV the inductive topology of all finite-dimensional subspaces. Then TV is already a strictly nuclear algebra. By the same procedure as above but now applied to TV one is lead to the algebra $\mathcal{P}(\mathbb{C}^n_q)$ of *polynomial functions on the quantum n-vector space*. $\mathcal{P}(\mathbb{C}^n_q)$ comprises a deformation of the algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n complex variables.

(ii) *Quantum exterior algebra* (cf. [8,12]). In the spirit of the preceeding examples it is also possible to deform the exterior algebra on $\mathbb{C}^n$. Let V' be the dual of V, $(\xi_1, \ldots, \xi_n)$ the dual basis of $(x_1, \ldots, x_n)$ and let TV' be given the finest lc topology. Then consider the closed ideal $J \subset \mathcal{O}(\mathbb{C}^*)\hat{\otimes}\mathrm{T}V'$ generated by the relations

$$\xi_\iota^2 = 0, \qquad \xi_\iota \xi_\kappa = -z^{-1}\xi_\kappa \xi_\iota \quad 1 \le \iota < \kappa \le n. \tag{9}$$

The corresponding quotient $\Lambda(\mathbb{C}^n_q) = \mathrm{T}V'/J$ is the *exterior algebra of the quantum n-vector space*. Exactly like above it is shown that $\Lambda(\mathbb{C}^n_q)$ is a holomorphic deformation over $\mathbb{C}^*$ of the exterior algebra $\Lambda(\mathbb{C}^n)$. Note that unlike $\mathcal{O}(\mathbb{C}^n_q)$ and $\mathcal{P}(\mathbb{C}^n_q)$ the algebra $\Lambda(\mathbb{C}^n)$ is finite-dimensional. The tensor product algebra $\mathcal{O}(\mathbb{C}^n_q) \otimes \Lambda(\mathbb{C}^n)$ can be interpreted as the algebra of *entire holomorphic quantum differential forms on* $\mathbb{C}^n_q$, the tensor product $\mathcal{P}(\mathbb{C}^n_q) \otimes \Lambda(\mathbb{C}^n)$ as the algebra of *algebraic quantum differential forms on* $\mathbb{C}^n_q$.

4. Holomorphic deformation of nuclear Hopf algebras

The concept of a (topologically free) holomorphic deformation can easily be transferred to the case of deformations of nuclear coalgebra structures, bialgebra and Hopf algebra structures, as well as Lie bialgebra structures. In particular a (*topologically free*) *holomorphic deformation* of a nuclear Hopf algebra H with structure maps μ, Δ, η, ε and S is given by the following data:

$$\begin{aligned}
&\tilde{\mu} \in \mathcal{O}(\Omega, \mathcal{L}(H\hat{\otimes}H, H)) \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}H_\Omega, H_\Omega)\\
&\quad \text{with } \tilde{\mu}(*) = \mu,\\
&\tilde{\Delta} \in \mathcal{O}(\Omega, \mathcal{L}(H, H\hat{\otimes}H)) \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}H_\Omega)\\
&\quad \text{with } \tilde{\Delta}(*) = \Delta,\\
&\tilde{\eta} \in \mathcal{O}(\Omega, \mathcal{L}(\mathbb{C}, H)) \cong \mathcal{L}_{\mathcal{O}(\Omega)}(\mathcal{O}(\Omega), H_\Omega) \quad \text{with } \tilde{\eta}(*) = \eta,\\
&\tilde{\epsilon} \in \mathcal{O}(\Omega, \mathcal{L}(H, \mathbb{C})) \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, \mathcal{O}(\Omega)) \quad \text{with } \tilde{\epsilon}(*) = \epsilon,\\
&\tilde{S} \in \mathcal{O}(\Omega, \mathcal{L}(H, H)) \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega) \quad \text{with } \tilde{S}(*) = S,
\end{aligned} \tag{10}$$

such that $(H_\Omega, \tilde{\mu}, \tilde{\Delta}, \tilde{\eta}, \tilde{\epsilon}, \tilde{S})$ is a nuclear Hopf algebra. Two such Hopf algebra deformations are *equivalent* if the corresponding Hopf algebra structures on H_Ω are isomorphic. Any holomorphic deformation of a nuclear Hopf algebra turns out to be equivalent to a holomorphic deformation with constant unit and counit, i.e. with $\tilde{\eta}(z) = \eta$ and $\tilde{\epsilon}(z) = \epsilon$ for all $z \in \Omega$.

Our first result concerns the dual of the holomorphic deformation $\tilde{H}$ of a strictly nuclear Hopf algebra H: Transposing all the structure maps one gets a holomorphic deformation of the dual nuclear Hopf algebra H'. The underlying topological $\mathcal{O}(\Omega)$-module is $H_\Omega{}' = \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, \mathcal{O}(\Omega))$ with the topology of uniform convergence on bounded (or equivalently compact) sets of $H_\Omega \cong \mathcal{O}(\Omega)\hat{\otimes}H$. This space is isomorphic as an $\mathcal{O}(\Omega)$-module to $\mathcal{O}(\Omega)\hat{\otimes}H' \cong \mathcal{O}(\Omega, H') = H'_\Omega$. The structure maps on H'_Ω are given by pullbacks of the structure maps on H_Ω:

$$\tilde{\Delta}' = \tilde{\mu}^*, \quad \tilde{\mu}' = \tilde{\Delta}^*, \quad \tilde{S}' = \tilde{S}^*, \quad \tilde{\eta}' = \tilde{\epsilon}^*, \quad \tilde{\epsilon}' = \tilde{\eta}^*. \tag{11}$$

Note that $(H_\Omega\hat{\otimes}H_\Omega)' \cong H'_\Omega\hat{\otimes}H'_\Omega \cong \mathcal{O}(\Omega, H'\hat{\otimes}H')$, hence $\tilde{\Delta}'$ is well-defined indeed. The following duality theorem now is obvious.

Theorem 4.1. *Let $\tilde{H} = (H_\Omega, \tilde{\mu}, \tilde{\Delta}, \tilde{\epsilon}, \tilde{\eta}, \tilde{S})$ be a holomorphic deformation of the strictly nuclear Hopf algebra $(H, \mu, \Delta, \epsilon, \eta, S)$. Then $\tilde{H}' = (H'_\Omega, \tilde{\mu}', \tilde{\Delta}', \tilde{\eta}', \tilde{\epsilon}', \tilde{S}')$ is a holomorphic deformation of the nuclear Hopf algebra H'. Moreover, $\tilde{H}$ and $\check{H}$ are equivalent if and only if $\tilde{H}'$ and $\check{H}'$ are equivalent. In addition, the bidual $(\tilde{H}')'$ is canonically isomorphic to $\tilde{H}$.*

The above theorem is a direct generalization of the corresponding result for the formal case (cf. [1]). Similarly, the next fact about the construction of a twisting matrix carries over directly to the holomorphic case (cf. [1, Proposition 4.2.4]).

Theorem 4.2. *Let G be a compact group and let H denote the strictly nuclear Hopf algebra $\mathcal{R}(G)' \cong \tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$. Let $(H_\Omega, \tilde{\Delta})$ be a holomorphic deformation of the nuclear bialgebra H leaving invariant the algebra product on H. Then there exists $\mathcal{F} \in (H\hat{\otimes}H)_\Omega$ such that $\tilde{\Delta}\mathcal{F} = \mathcal{F}\Delta$.*

Proof. Because of $\tilde{\Delta} \in \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega \otimes_{\mathcal{O}(\Omega)} H_\Omega) \cong \mathcal{O}(\Omega, \mathcal{L}(H, H\hat{\otimes}H))$ we get a "twisting matrix" $\mathcal{F} \in \mathcal{O}(\Omega, H\hat{\otimes}H) \cong (H\hat{\otimes}H)_\Omega$ by the integral $\mathcal{F} = \int_G \tilde{\Delta}(g)(\Delta(g))^{-1}\,\mathrm{d}\mu(g)$, where μ is the Haar measure on G and $g \in G$ stands for δ_g, the Dirac distribution with support in g. By left- and right-invariance of the Haar measure

$$\mathcal{F}\Delta(h) = \int_G \tilde{\Delta}(hg)(\Delta(hg))^{-1}\Delta(h)\,\mathrm{d}\mu(hg) = \tilde{\Delta}(h)\mathcal{F}. \tag{12}$$

Hence the claim follows from the fact that the Dirac distributions δ_g lie densely in $\mathcal{R}(G)'$. □

Remark 4.3. We expect in all important examples $\mathcal{F}(z)$ to be invertible at least in a neighborhood of the base point. In that case we get a dual version of Theorem 4.2 which leads to a holomorphic coquasitriangular Hopf algebra deformation of $\mathcal{R}(G)$.

Theorem 4.4. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a cocommutative nuclear Hopf algebra and let $\mathcal{F} : \Omega \to H\hat{\otimes}H$ be a continuous map on an open domain $\Omega \subset \mathbb{C}$ fulfilling the axioms of a twisting map (cf. [2]). Then $\mathcal{F}$ induces a new nuclear Hopf algebra structure $(H_\Omega, \mu^{\mathcal{F}}, \eta^{\mathcal{F}}, \Delta^{\mathcal{F}}, \varepsilon^{\mathcal{F}}, S^{\mathcal{F}})$ on $H_\Omega = \mathcal{O}(\Omega)\hat{\otimes}H$ by defining $\mu^{\mathcal{F}} = \mu, \eta^{\mathcal{F}} = \eta$, $\Delta^{\mathcal{F}}(z \otimes h) = \mathcal{F}(z)\Delta(h)\,\mathcal{F}^{-1}(z)$, $\varepsilon^{\mathcal{F}} = \varepsilon$ and $S^{\mathcal{F}}(z \otimes h) = v(z)S(h)v^{-1}(z)$ with $z \in \mathcal{O}(\Omega)$, $h \in H$ and $v = \mu(1 \otimes S)\mathcal{F}$. $(H_\Omega, \mu, \eta, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is topologically triangular with universal R-matrix $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.*

Now assume H to be topologically quasitriangular with universal R-matrix $\mathcal{R}$ and, additionally to the above, that $\mathcal{F}$ fulfills the quantum Yang–Baxter equation and $\mathcal{F}_{21} = \mathcal{F}^{-1}$. Then $(H_\Omega, \mu, \eta, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a topologically quasitriangular nuclear Hopf algebra with universal R-matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}^{-1}\mathcal{R}\mathcal{F}^{-1}$.

If the twisting matrix $\mathcal{F}$ fulfills $\mathcal{F}(*) = 1 \otimes 1$ for a point $* \in \Omega$, then in both of the above cases $(H_\Omega, \mu, \eta, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ comprises a topologically free holomorphic deformation over Ω of the Hopf algebra H with base point $*$.

Proof. The proof of the theorem can be taken almost literally from the corresponding one in the formal case. Confer for example [2]. □

In the following considerations we will show how one can construct under certain conditions a holomorphic deformation of an algebra (resp. bialgebra or Hopf algebra) out of a formal one.

So let A be a finitely generated $\mathbb{C}$-algebra and assume that there exists a formal deformation $A_\hbar \cong (A[[\hbar]], \mu, \eta)$ of A. In other words μ is a $\mathbb{C}[[\hbar]]$-bilinear multiplication map on $A[[\hbar]]$ and η a unit such that $A[[\hbar]]/\hbar A[[\hbar]] \cong A$. We now choose a finite-dimensional vector space V and a surjective homomorphism $\mathrm{T}V = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} \to A$. According to (2) this gives rise to a topological presentation $\hat{\mathrm{T}}V \to \hat{A}$, where $\hat{A}$ and $\hat{\mathrm{T}}V$ are the completions of A resp. $\mathrm{T}V$ with respect to the topology of all finite-dimensional representations (resp. all nuclear Fréchet representations). Using the universal property of the tensor algebra $\mathrm{T}V$ there exists a unique morphism of $\mathbb{C}[[\hbar]]$-algebras $\pi : \mathrm{T}V[[\hbar]] \to A_\hbar$ such that

$\mathrm{T}V \to \mathrm{T}V[[\hbar]] \to A_\hbar \to A$ is the presentation $\mathrm{T}V \to A$. Let I be the kernel of π and $\hat{I}$ its completion in $\hat{\mathrm{T}}V[[\hbar]] = \mathbb{C}[[\hbar]]\hat{\otimes}\hat{\mathrm{T}}V$. Denoting the nuclear algebra $\hat{\mathrm{T}}V[[\hbar]]/\hat{I}$ by $\hat{A}_\hbar$ we then have a commutative diagram

$$\begin{array}{ccc} \mathrm{T}V[[\hbar]] & \xrightarrow{\ \iota\ } & \hat{\mathrm{T}}V[[\hbar]] \\ \pi \downarrow & & \downarrow \hat{\pi} \\ A_\hbar & \xrightarrow[\hat{\iota}]{} & \hat{A}_\hbar \end{array}$$

with injective horizontal and surjective vertical arrows. Only the injectivity of $\hat{\iota}$ is not immediately clear. It follows from the fact that I is closed in $\mathrm{T}V[[\hbar]]$: every image of I under a projection of $\mathrm{T}V[[\hbar]]$ to a finite-dimensional vector space is closed. Note that the morphisms in the above diagram are also filtered with respect to the filtrations induced by the maximal ideals generated by $\hbar$. By these considerations we now have $\hat{A}_\hbar/\hbar\hat{A}_\hbar \cong \hat{A}$. Further the vector space $A[[\hbar]] \cong A_\hbar$ lies densely in $\hat{A}_\hbar$, hence $\hat{A}_\hbar \cong \hat{A}[[\hbar]]$ and the algebra $\hat{A}_\hbar$ comprises a formal deformation of $\hat{A}$.

In the next step assume Ω to be a connected open domain in $\mathbb{C}$ containing the origin. This gives us a continuous and injective map $\rho : \hat{\mathrm{T}}V_\Omega = \mathcal{O}(\Omega)\hat{\otimes}\hat{\mathrm{T}}V \to \hat{\mathrm{T}}V[[\hbar]]$ by power series expansion around the origin. We now say that the formal deformation $A_\hbar$ has *holomorphic initial data* if there exists a system of generators Y of I such that $Y \in \mathrm{im}(\rho)$. Assuming this is the case indeed let $\tilde{Y}$ be the preimage of Y under ρ. Let I_Ω be the closed ideal in $\hat{\mathrm{T}}V_\Omega$ generated by $\tilde{Y}$. We then have $I_\Omega = \rho^{-1}(\hat{I})$, which induces an injective and filtered homomorphism $\tilde{A} := \mathcal{O}(\Omega)\hat{\otimes}\hat{\mathrm{T}}V/I_\Omega \to \hat{\mathrm{T}}V[[\hbar]]/\hat{I} \cong \hat{A}[[\hbar]]$. This map is surjective from $\mathfrak{m}^n\tilde{A}/\mathfrak{m}^{n+1}\tilde{A}$ to $\hbar^n\hat{A}[[\hbar]]/\hbar^{n+1}\hat{A}[[\hbar]]$ where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}(\Omega)$ of functions vanishing at the origin. Thus $\tilde{A}/\mathfrak{m}\tilde{A} \cong \hat{A}$. As furthermore $\tilde{A}$ is dense in $\hat{A}[[\hbar]]$ we finally have $\tilde{A} \cong \hat{A}_\Omega$ as a nuclear space. Thus $\tilde{A}$ is a topologically free holomorphic deformation of $\hat{A}_\Omega$. We subsume these results in the following proposition.

Proposition 4.5. *Let A be a finitely generated complex algebra (resp. bialgebra or Hopf algebra) and let $\hat{A}$ be the completion of A with respect to the topology of finite-dimensional representations (resp. nuclear Fréchet representations). Then every formal deformation $A_\hbar$ of A induces a formal deformation $\hat{A}_\hbar$ of $\hat{A}$ together with a canonical filtered embedding $A_\hbar \to \hat{A}_\hbar$. If the deformation $A_\hbar$ has holomorphic initial data over some open connected domain $\Omega \subset \mathbb{C}$ containing the origin, then there exist a topologically free holomorphic deformation $\tilde{A}$ of $\hat{A}$ over Ω and a canonical filtered embedding $\tilde{A}_\Omega \to \hat{A}_\hbar$. These constructions are unique up to isomorphy.*

Remark 4.6. Suppose $H_\hbar$ to be a formal deformation of a finitely generated Hopf algebra H with holomorphic initial data. Give $H_\hbar$ the projective limit topology with respect to all projections $H_\hbar \to H_\hbar/\hbar^m H_\hbar \cong H^m \subset \hat{H}^m$. Assume further that $H_\hbar$ is topologically (quasi-) triangular with universal R-matrix $\mathcal{R}_\hbar \in H_\hbar\hat{\otimes}H_\hbar$. The universal R-matrix on $H_\hbar$

can be pushed down to one on $\tilde{H}_\Omega$ or in other words induces on $\tilde{H}_\Omega$ the structure of a topologically (quasi-) triangular Hopf algebra, if and only if the formal power series expansion of $\mathcal{R}_\hbar$ with respect to $\hbar$ in $\hat{H}\hat{\otimes}\hat{H}$ is converging over Ω.

Finally let us show by giving an example that the theory of quantum groups can be understood as the theory of holomorphic deformations.

Quantized $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$. Consider the Lie algebra $\mathfrak{sl}(N+1,\mathbb{C})$. Using its standard basis $X_\iota^+, X_\iota^-, H_\iota$ with $1 \le \iota \le N$ one checks immediately that the relations defining quantized $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$ are holomorphic in the sense of Proposition 4.5. Hence, we receive a holomorphic deformation $\hat{\mathcal{U}}_q\mathfrak{sl}(N+1,\mathbb{C})$ of $\hat{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$ over the domain $\Omega = \{z \in \mathbb{C}: z \neq k\pi\mathrm{i},\ k \in \mathbb{Z}^*\}$. Using the Drinfeld double (cf. [3]) one can construct a (topological) R-matrix $\mathcal{R}_\hbar$ on formally quantized $\mathcal{U}_{\mathfrak{sl}}(N+1,\mathbb{C})$ such that $\mathcal{R}_\hbar$ has an expansion of the form

$$\mathcal{R}_\hbar = \sum_{\beta\in\mathbb{N}^N} \left(\exp\left(\hbar\left[\frac{1}{2}t_0 + \frac{1}{4}(H_\beta \otimes 1 + 1 \otimes H_\beta)\right]\right)\right) P_\beta, \tag{13}$$

where $t_0 \in \mathfrak{sl}(N+1,\mathbb{C}) \otimes \mathfrak{sl}(N+1,\mathbb{C})$ is chosen appropriate, $H_\beta = \sum_\iota \beta_\iota H_\iota$ and the P_β are polynomials homogeneous of degree β_ι in $X_\iota^+ \otimes 1$ and $1 \otimes X_\iota^-$. Hence $\mathcal{R}_\hbar$ has a converging power series expansion, so Remark 4.6 entails that $\mathcal{U}_q\mathfrak{sl}(N+1,\mathbb{C})$ is topologically quasitriangular as well.

Quantizing $\mathrm{SL}(N,\mathbb{C})$ *according to FRT.* First let us briefly recall the Faddeev–Reshetikhin–Takhtajan-construction of quantized $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ (cf. [4]). Let V be the complex vector space spanned by x_i, $i = 1,\ldots,N$, $C = \mathrm{End}(V)'$ and t_i^j, $i, j = 1,\ldots,N$ the basis of C induced by the x_i. Then consider the following R-matrix

$$\begin{aligned} &R : \mathbb{C}^* \to C \otimes C, \\ &z \mapsto z\sum_{i=1}^{n} t_i^i \otimes t_i^i + \sum_{\substack{i,j=1\\ i\neq j}}^{n} t_i^i \otimes t_j^j + (z - z^{-1}) \sum_{\substack{i,j=1\\ i>j}}^{n} t_i^j \otimes t_j^i \end{aligned} \tag{14}$$

which by the FRT-construction gives rise to the quotient algebra $A(R) = TC/J$. Hereby TC is the tensor algebra of C together with the finest lc topology, and J the closed ideal in TC generated by the RTT-relations (cf. [4]). Now it is well-known that the *quantum determinant* $\det_q T = \sum_{\sigma\in S_N} (-z)^{\ell(\sigma)} t_1^{\sigma(1)} \cdot \ldots \cdot t_N^{\sigma(N)}$ belongs to the center of $A(R)$. Denoting by I the ideal $I = A(R)(\det_q T - 1)$ the quotient bialgebra $A(R)/I$ then is even a Hopf algebra. We call it the *algebra of matrix coefficients on quantized* $\mathrm{SL}(N,\mathbb{C})$ and denote it by $\mathcal{R}(\mathrm{SL}_q(N,\mathbb{C}))$. Furthermore denote for every $z \in \mathbb{C}^*$ by $\mathcal{R}(\mathrm{SL}_z(N,\mathbb{C}))$ the Hopf algebra $\mathcal{R}(\mathrm{SL}_q(N,\mathbb{C}))/\mathfrak{m}_z\mathcal{R}(\mathrm{SL}_q(N,\mathbb{C}))$, where $\mathfrak{m}_z$ is the maximal ideal of $\mathcal{O}(\mathbb{C}^*)$ at z. Likewise define $\mathcal{U}_z\mathfrak{sl}(N,\mathbb{C})$ for $z \in \Omega = \{z \in \mathbb{C} : z \neq k\pi\mathrm{i}, k \in \mathbb{Z}^*\}$.

Then the following result is an immediate consequence of Proposition 4.5 and the corresponding result in the formal case (cf. [2, 7.1]).

Theorem 4.7. *The FRT-algebra* $\mathcal{R}(\mathrm{SL}_q(N,\mathbb{C})) = A(R)/(\det_q - 1)A(R)$ *corresponding to the R-matrix* (14) *comprises a holomorphic quantization of the Poisson algebra*

$\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ *of matrix coefficients on the Lie group* $\mathrm{SL}(N,\mathbb{C})$. *Moreover, it coincides with the deformation* $\mathcal{U}_q\mathfrak{sl}(N,\mathbb{C})'$ *dual to the quantization of* $\mathcal{U}\mathfrak{sl}(N,\mathbb{C})$. *For every* $z \in \Omega$ *the Hopf algebra* $\mathcal{R}(\mathrm{SL}_{e^z}(N,\mathbb{C}))$ *is topologically isomorphic to the restricted Hopf dual* $\mathcal{U}_z\mathfrak{sl}(N,\mathbb{C})^\circ$, *and* $\mathcal{U}_z\mathfrak{sl}(N,\mathbb{C})$ *to* $\mathcal{R}(\mathrm{SL}_{e^z}(N,\mathbb{C}))'$.

Acknowledgements

An essential part of this work has been done while both of the authors participated for two weeks in the *Research in Pairs* program at the Forschungsinstitut Oberwolfach. We thank the staff at Obserwolfach for the excellent working conditions and greatly acknowledge financial support by the Stiftung Volkswagenwerk.

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